The Prehistory of Conformal Mapping

Development of Mathematical Cartography in the Eighteenth Century

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The stereographic projection, used in ancient times, and Mercator's projection, developed in the sixteenth century, were early examples of conformal, or angle-preserving projections. The term *proiecto conformalis* was not introduced until 1789 ([Youschkevitch, 1991], p. 746), but, as we shall see, the ideas were in place with the work of Lambert about fifteen years prior.

1 The Stereographic Projection

The stereographic projection, or planisphere, may have been known in ancient Egypt [Keuning, 1955]. Synesius of Cyrene attributes its discovery to Hipparchus ([Heath, 1981], p. 293). Ptolemy's *Planisphaerum* describes its use, but in astronomy, not geography; in fact this projection seems to have been used exclusively for star charts and astrolabes until the Renaissance.

The planisphere has two appealing geometric properties. First, all circles on the sphere (both great circles and small circles) are carried into circles (or exceptionally into straight lines) on the plane. This property, a simple consequence of Apollonius' work on subcontrary cutting planes (e.g., Book I, Prop. 5, of the *Conics*), was apparently known to Ptolemy, although he gave no general proof.

About 1500, the stereographic function began to be widely used and popular for geographic maps [Keuning, 1955]. Lagrange [Lagrange, 1778] stated that "most modern Geographers" used it to construct their maps. This may have been because it was so easy to construct. Since all meridians and parallels were circles, one needed to find only (as Lagrange pointed out) three points on each to trace its entire curve.

The second property of the planisphere, apparently unknown to Ptolemy, is that the projection preserves angular measure. That is, if two great circles on the sphere intersect at a given angle, then their images on the plane (which, by the first property, are circles or lines) intersect at the same angle. The fact can be proved without calculus, and is left as an exercise for the reader, with the following hints:

1. Without loss of generality, assume that one of the two great circles is a meridian. The other great circle is either a meridian (in which case the problem is trivial) or is inclined at an angle ϕ to the meridian.

2. The image of the second great circle is a planar circle C whose radius is $\csc \phi$.

3. the distance of the center of C from the origin on the plane (the image of the pole) is $\cot \phi$.

2 Mercator

As the compass came into use in the late Middle Ages, ship captains began sailing out of sight of coastal landmarks, navigating by keeping their ships on a constant compass bearing (a constant angle between the ship's direction and the local meridian). Unless the direction is straight north, south, east, or west, the curve (called a *rhumb line* or *loxodrome*) traced out by a ship on a constant compass heading is represented on the globe by a line which spirals logarithmically toward each pole.

A loxodrome is not a great circle. Pedro Nuñez proved this in 1537 in his *Tratado de Esphera.* We do not know whether the Flemish mapmaker Gerard Mercator ever heard about the works of the Portuguese geometer. But by 1541, Mercator found a way to draw accurate loxodromes on a globe, and, eventually, to transfer them to a straight line on a sheet of paper. In 1569, he published his famous world map, with the Latin title *A new and enlarged description of the earth, with improvements for use in navigation.*

We do not know how Mercator actually constructed the map. He certainly used graphic instruments, since tables of secants were not yet available. One plausible theory, advanced by Nordenskiöld [Keuning, 1955], was that Mercator divided the surface of the globe into zones of latitude ten degrees wide. While (graphically) transferring each zone to the plane, he scaled the length so that it matched the length of the equator, and then scaled the width of the zone by the same amount. On a globe of unit equatorial circumference, this would mean that a zone about the circle of parallel at latitude ϕ would need to be enlarged by $\sec\phi$, and the parallel would be represented by a straight line at constant ordinate

$$0.5 \cdot \sec\phi \cdot z + \sum_{\psi<\phi} \sec\psi \cdot z,$$

where z represents the width of each zone. Mercator used $z = 10°$; by taking limits as z goes to zero, we get the Mercator projection M from a point iat latitude ϕ, longitude λ on the sphere as

$$M_x(\phi, \lambda) = \lambda, \qquad M_y(\phi, \lambda) = \int_0^\phi \sec\phi \, d\phi.$$

(Of course, on a real map, the planar coordinates M_x, M_y would be scaled and shifted in some appropriate fashion.)

We close this section with a curiosity worth mentioning, although it would not have been recognized until the nineteenth century. If, like Riemann, we view the codomain of the stereographic projection S as the Argand plane of complex numbers, then (projecting from the north pole to the plane tangent at the south pole)

$$S(\phi, \lambda) = \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \cdot \exp(i\lambda).$$

The Mercator projection becomes

$$\begin{aligned} M(\phi, \lambda) &= \lambda + i \cdot \int_0^\phi \sec\psi \, d\psi \\ &= \lambda + i \cdot \ln\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right), \end{aligned}$$

so that (modulo a reflection through the real axis), the Mercator projection is i times the complex logarithm of the stereorgraphic projection.

3 Lambert

Sailors gradually discovered the advantages of Mercator's projection and came to use it exclusively for their nautical charts (in fact it is still used today). However, it was apparently not until the early eighteenth century that serious attention focussed on the problems of terrestrial cartography. Perhaps this came about in part because of the foundation of modern nation-states, with their need to map large areas acccurately, both for taxation and for warfare. The invention of the telescope made accurate surveying possible; also, by observation of the moons of Jupiter, accurate measurements of longitude (on land, at any rate) became possible. As long sea voyages became common, travelers began to report that pendulum clocks consistently ran slow near the equator. Newton suggested that perhaps the earth was not quite spherical; that it bulged slightly at the equator and thus the force of gravity was less there. The theory was confirmed after a French expedition to Ecuador in the 1730s.

Johann Heinrich Lambert published his *Anmerkungen und Zusätze zur Entwerfung der Land und Himmelscharten* in 1772. This strikingly original work is considered the foundation of modern mathematical cartography. Lagrange gives Lambert credit as the first to characterize the problem of mapping from a sphere to a plane, while preserving some given property, in terms of nonlinear partial differential equations. The technique was astoundingly fruitful, for he invented several whole families of conformal and equal-area projections, some of which are still in widespread use today. Lambert also seems to have been the first to take account of the ellipsoidal, rather than spherical, shape of the earth.

We shall look in some detail at his development of a projection now called Lambert's Conformal Conic, then briefly examine another projection called the Transverse Mercator.

In a section subtitled "More General Method to Represent the Spherical Surface so that All Angles Preserve their Size," Lambert begins:

> "Stereographic representations of the spherical surface, as well as Mercator's nautical charts, have the peculiarity that all angles maintain the sizes that they have on the surface of the globe. This yields the greatest similarity that any plane figure can have with on drawn on the surface of a sphere. The question has not been asked whether this property occurs only in the two methods of representation mentioned, or whether these two representations, so different in appearances, can be made to approach each other through intermediate stages. Mercator represents the meridians as parallel lines, perpendicular to the equator... Contrastingly, in

> the polar case of the stereographic projection, the same straight meridians intersect at the proper angle. Consequently, if there are stages intermediate to these two representations, they must be sought by allowing the angle of intersections of the meridians to be arbitrarily larger or smaller than its value on the surface of the sphere. This is the way in which I shall now procede."

In other words, Lambert proposes to investigate conic projections. These have a history dating back at least to Ptolemy's *Geography*, which describes (more or less) an "equidistant" conic projection, in which all the parallels of latitude are equally spaced. Conic projections preserve scale along parallels of latitude, and thus are well suited for regions of the globe which extend east-west rather than north-south. In the eighteenth century, the equidistant conic was apparently rediscovered by Nicolas DeLisle, Euler's colleague at St. Petersburg, and used for the 1745 atlas of Siberia.

Lambert considers the representation M on the plane of a spherical point with colatitude ϵ. Let P be the representation of the pole, μ be a point at the same latitude as M, such that the angle $MP\mu$ is "infinitely small." Since this is an angle between the representations of two meridians, and since there is assumed to be a constant ratio m between the angles of meridians on the sphere and their representations on the plane, we can also write this angle as m times the difference in longitudes $d\lambda$. Finally, let N be the representation of a point on the same meridian as M but with colatitude $\epsilon + d\epsilon$, and ν be the point which completes the trapezoid.

If $\mu MN\nu$ is to be similar to the spherical figure it represents, then it must preserve the same proportion of sides:

$$\frac{\mu M}{MN} = \frac{d\lambda \sin\epsilon}{d\epsilon},$$

or, if we set $PM = x$, $MN = dx$, so that

$$M\mu = x \cdot \angle MP\mu = x\, m\, d\lambda,$$

then (3) becomes

$$\frac{m\, x\, d\lambda}{dx} = \frac{d\lambda \sin\epsilon}{d\epsilon}, \qquad \text{or} \qquad \frac{dx}{x} = \frac{m\, d\epsilon}{\sin\epsilon}.$$

Lambert now integrates both sides to obtain

$$\ln x = m \ln \tan \frac{1}{2}\epsilon.$$

[Tobler(tr), 1972], p. 28

If we now assume that x takes the value 1 when M represents a point on the equator (this is only a matter of scale), then the constant C must be zero, so that

$$x = \left(tan\frac{\epsilon}{2}\right)^m.$$

Lambert observes that the case $m = 1$ corresponds to the stereographic projection. By rewriting (3) in terms of the latitude ϕ rather than the colatitude ϵ, he obtains:

$$x = \tan^m\left(\frac{\pi}{4} - \frac{\phi}{2}\right) = \left(\frac{1 - \tan\frac{\phi}{2}}{1 + \tan\frac{\phi}{2}}\right)^m,$$

Using the (infinite) binomial expansions of both numerator and denominator, he obtains, (in modern notation):

$$x = \left(1 - m\tan\frac{\phi}{2} + m\frac{m-1}{2}\tan^2\frac{\phi}{2} - \cdots\right) \cdot \left(1 - m\tan\frac{\phi}{2} + m\frac{m+1}{2}\tan^2\frac{\phi}{2} - \cdots\right)$$

$$= 1 - 2m\left(\tan\frac{\phi}{2} + \frac{1}{3}\tan^3\frac{\phi}{2} + \frac{1}{5}\tan^5\frac{\phi}{2} + \cdots\right) + o(m^2),$$

and from this he shows that the expression approaches the Mercator projection as m approaches 0.

Lambert goes on to consider another family of conformal projections in which both meridians and parallels are represented as circular arcs. He then attempts to give the most general possible solution to the conformal condition. He considers the following system:

$$dy = M\,d\phi + m\,d\lambda$$
$$dx = N\,d\phi + n\,d\lambda$$

where M, N, m, and n are unknown functions of ϕ and λ. Geometric considerations, similar to those above, lead to the two conditions

$$M\cos\phi = n, \quad -N\cos\phi = m.$$

Lambert shows that the Mercator and stereographic projections fit this framework, but is unable to find a general closed-form solution. He mentions a method of solution which he credits to Lagrange. But "since one eventually relies on infinite series using this procedure, I return directly to the two differential equations." Finally, he gives y as the doubly infinite series:

$$\begin{aligned} y = A & \quad +B\lambda & \quad +C\lambda^2 + \cdots \\ + A' \sin\phi & \quad +B'\lambda \sin\phi & \quad +C'\lambda^2 \sin\phi + \cdots \\ + A'' \sin 2\phi & +B''\lambda \sin 2\phi & +C'' \sin\lambda^2 \sin 2\phi + \cdots \\ + A''' \sin 3\phi & +B'''\lambda \sin 3\phi & +C''' \sin\lambda^2 \sin 3\phi + \cdots \\ + \cdots & & \end{aligned}$$

and x as a doubly infinite series in terms of $\cos k\phi$. Lambert computes more than thirty of the coefficients, and gives an explicit formula for finding (recursively) all coefficients of terms involving the first five powers of λ.

The prodigious labor pays off when Lambert looks for a conformal map on which the equator and central meridian are both straight lines, and parallels intersect the central meridian at equal intervals. By manipulating the infinite series, and then "summing from below," he finds

$$\begin{aligned} x &= \phi + \arctan \frac{\sin 2\phi \tan^{\frac{1}{2}} \lambda}{1 - \cos 2\phi tan^2 \frac{1}{2}\lambda} \\ y &= \frac{1}{2} \ln \left(\frac{1 + 2\tan\frac{1}{2}\lambda \cos\phi + \tan^2 \frac{1}{2}\lambda}{1 - 2\tan\frac{1}{2}\lambda \cos\phi + \tan^2 \frac{1}{2}\lambda} \right) \end{aligned}$$

Not simple, perhaps, but at least in closed form. This is the sideways, or transverse, version of the Mercator projection, in which the circle of true scale is a meridian rather than the equator. It does not map loxodromes to straight lines, and most circles on the globe become transcendental curves. Nevertheless, it has turned out to be enormously practical for maps of small regions, both because of good error control, and because adjacent regions north to south can be tiled together. The "Transverse Mercator", with adjustments, also given by Lambert, for the ellipsodial shape of the earth, is today the most widely used projection for large scale maps in the United States.

Afterword

This paper was written with the hope that it might be useful to two audiences: those who will not read the original authors, but want a summary of the ideas, and those who do want to read the originals, and would like a guide to the main ideas. To the extent possible, I have tried to follow the original notation, with some minor concessions to make it easier on the modern reader, e.g. x^2 in place of xx, "ln" in place of "log", and so forth. The illustrations are copies of those in the original works.

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